The Fundamental Theorem of Calculus

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5/5/08

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1. Introduction

If you have ever taken a course in calculus, you have probably been introduced to the Fundamental Theorem of Calculus. This theorem states that, if f is continuous on the interval [a,b] and F is the antiderivative of f on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The theorem that I will prove is Stokes' Theorem, which is sometimes called the Fundamental Theorem of Calculus in higher dimensions.

2. Stokes' Theorem

Stokes' Theorem states that if ω is a (k-1)-form on an open set A $\subset R^n$ and c is a k-chain in A, then

$$\int_c dw = \int_{\partial c} w$$

If k=1 and c is a function from [0,1] to R, we can see that Stokes' Theorem is just our old friend, the Fundamental Theorem of Calculus. In general, however, the applications of Stokes' Theorem are more complex.

3. Proof

We will suppose that c=I^k, where I^k is a function from $[0,1]^k \to R^k$ defined by I^k(x)=x for x $\in [0,1]^k$. We will also suppose that ω is a (k-1)-form on $[0,1]^k$. Then we can view ω as the sum of (k-1)-forms that look like

$$f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k,$$

where dx^i is omitted. We can prove Stokes' Theorem for each of these forms.

Thus, on the right side of the equation, $\int_{\partial c} w$, we now have

$$\int_{\partial I^k} f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k.$$

By applying the definition of the differential, we obtain:

$$\sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{I^k_{(j,\alpha)}} f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k.$$

This is equal to

$$\sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} (I^k)_{(j,\alpha)} * (f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k).$$

Now

$$\int_{[0,1]^{k-1}} (I^k)_{(j,\alpha)} * (f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

is equal to

$$\int_{[0,1]^{k-1}} f \circ I^k_{(j,\alpha)} \cdot I^k_{(j,\alpha)} * (dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

when we apply the definition of the * operation. This is equal to

$$\int_{[0,1]^{k-1}} f \circ I^k_{(j,\alpha)} (I^k_{(j,\alpha)} * dx^1) \wedge \cdots \wedge (\widehat{I^k_{(j,\alpha)} * dx^i}) \wedge \cdots \wedge (I^k_{(j,\alpha)} * dx^k).$$

Now we can look at one of the individual elements $I^k_{(j,\alpha)} * dx^j$. By the definition of the * operation, this is equal to $\sum_{r=1}^{n} D_r (I^k_{(j,\alpha)})^j dx^r$. $I^k_{(j,\alpha)}$ is a function from $[0,1]^k \to R^k$. $(I^k_{(j,\alpha)})^j = (I^k(x^1, \cdots, x^{j-1}, \alpha, x^j, \cdots, x^{k-1}))^j$. This is simply equal to α, because I is the identity function and α is the jth term, if j $\neq$ i. Thus, $\sum_{r=1}^{n} D_r (I^k_{(j,\alpha)})^j dx^r$=0, since α is simply a constant. So

$$\int_{[0,1]^{k-1}} (I^k)_{(j,\alpha)} * (f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k) = 0.$$

But if j=i, then, since the omitted term in our original form is in the ith place, α will not be in the jth place; instead, there will be an x term in the jth place. In this case,

$$\int_{[0,1]^{k-1}} (I^k)_{(j,\alpha)} * (f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k) =$$

$$\int_{[0,1]^k} f(x^1, \cdots, \alpha, \cdots, x^k) dx^1 \cdots dx^k.$$

Thus,

$$\sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{I^k_{(j,\alpha)}} f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k =$$

$$(-1)^{i+1} \int_{[0,1]^k} f(x^1, \cdots, 1, \cdots, x^k) dx^1 \cdots dx^k + (-1)^i \int_{[0,1]^k} f(x^1, \cdots, 0, \cdots, x^k) dx^1 \cdots dx^k.$$

Now we will look at the left side of the equation for Stokes' Theorem.

$$\int_c dw = \int_{I^k} d(f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k).$$

By Fubini's Theorem, we can take the partial derivatives to obtain:

$$(-1)^{i-1} \int_0^1 \cdots (\int_0^1 D_i f(x^1, \cdots, x^k) dx^i) dx^1 \cdots \widehat{dx^i} \cdots dx^k.$$

Now by the Fundamental Theorem of Calculus (in just one dimension), we have

$$(-1)^{i-1} \int_0^1 \cdots \int_0^1 [f(x^1, \cdots, 1, \cdots, x^k) - f(x^1, \cdots, 0, \cdots, x^k)] dx^1 \cdots \widehat{dx^i} \cdots dx^k =$$

$$(-1)^{i+1} \int_{[0,1]^k} f(x^1, \cdots, 1, \cdots, x^k) dx^1 \cdots dx^k + (-1)^i \int_{[0,1]^k} f(x^1, \cdots, 0, \cdots, x^k) dx^1 \cdots dx^k.$$

Thus,

$$\int_{I^k} dw = \int_{\partial I^k} w. \qquad (1)$$

By definition, if c is a singular k-cube (which is a continuous function from $[0,1]^k \rightarrow$ A $\subset R^n$) then

$$\int_{\partial c} w = \int_{\partial I^k} c * w. \qquad (2)$$

Also by definition,

$$\int_c dw = \int_{I^k} c * (dw).$$

By the definition of the * operation, this is equal to

$$\int_{I^k} d(c * w) = \int_{\partial I^k} c * w$$

(by (1)). And by (2) we know that this is equal to

$$\int_{\partial c} w.$$

Thus,

$$\int_c dw = \int_{\partial c} w. \qquad (3)$$

Finally, if c is a sum of singular n-cubes with integer coefficients, we call c a k-chain $\sum a_i c_i$. Then we have

$$\int_c dw = \sum a_i \int_{c_i} dw.$$

By (3),

$$\sum a_i \int_{c_i} dw = \sum a_i \int_{\partial c_i} w = \int_{\partial c} w.$$

Thus, for any k-chain c,

$$\int_c dw = \int_{\partial c} w.$$